

Ricci flows and expansion in axion-dilaton cosmology

Ioannis Bakas

*Department of Physics, University of Patras,
26500 Patras, Greece
E-mail: bakas@ajax.physics.upatras.gr*

Domenico Orlando^{ab} and Panagiotis Marios Petropoulos^{ac}

^a*Centre de Physique Théorique, École Polytechnique,*
91128 Palaiseau, France*

^b*Theoretische Natuurkunde, Vrije Universiteit Brussel, and
The International Solvay Institutes,
Pleinlaan 2, 1050 Brussels, Belgium*

^c*Institut de Physique, Université de Neuchâtel,
Breguet 1, 2000 Neuchâtel, Switzerland
E-mail: domenico.orlando@cpt.polytechnique.fr,
marios@cpt.polytechnique.fr*

ABSTRACT: We study renormalization-group flows by deforming a class of conformal sigma-models. We consider overall scale factor perturbation of Einstein spaces as well as more general anisotropic deformations of three-spheres. At leading order in α' , renormalization-group equations turn out to be Ricci flows. In the three-sphere background, the latter is the Halphen system, which is exactly solvable in terms of modular forms. We also analyze time-dependent deformations of these systems supplemented with an extra time coordinate and time-dependent dilaton. In some regimes time evolution is identified with renormalization-group flow and time coordinate can appear as Liouville field. The resulting space-time interpretation is that of a homogeneous isotropic Friedmann–Robertson–Walker universe in axion-dilaton cosmology. We find as general behaviour the superposition of a big-bang (polynomial) expansion with a finite number of oscillations at early times. Any initial anisotropy disappears during the evolution.

KEYWORDS: Conformal Field Models in String Theory, Renormalization Group.

*Unité mixte du CNRS et de l'École polytechnique, UMR 7644.

Contents

1. Introduction	1
2. The renormalization-group-flow approach	4
2.1 The behaviour of the overall scale in general situations	4
2.2 Generic perturbations of S^3 : the Halphen system	5
3. Space-time solutions	11
3.1 Equations of motion for the dilaton and the scale factor	11
3.2 Linearization	13
3.3 The meaning of \bar{Q}	15
3.4 The anisotropic deformations of the three-sphere	16
4. Cosmological interpretation	17
5. Summary	21
A. The Ricci tensor for the deformed group manifold	22
B. The Jacobi functions	23

1. Introduction

Finding string backgrounds with cosmological interpretation is a challenging problem that has led to many developments, either in genuine string theory or in string-inspired models. One possible way to investigate string cosmology is by promoting a Euclidean three-dimensional static string solution to a Minkowskian four-dimensional time-dependent one. To some extent, this is borrowed from general relativity, where e.g. a three-sphere can be promoted to a Friedmann-Robertson-Walker (FRW) universe. The latter is a four-dimensional space-time which has spherical spatial section of time-dependent radius. However, string theory is more constraint than general relativity and such a promotion is a delicate issue, which does not admit a systematic treatment: the string equations of motion involve many fields (metric, dilaton and antisymmetric tensors) and the four-dimensional universe must necessarily be accompanied by an internal, compact space that saturates the central charge without spoiling the perturbative (in α') nature of the solution. It is not surprising therefore that the emergence of e.g. de Sitter solution in the context of strings is still unclear, whereas this universe is the archetype of a cosmological background in general relativity.

One can be more concrete and give a specific example to illustrate the above difficulties. A three-sphere background appears in string theory as the target space of a Wess-Zumino-Witten (wzw) model. This sigma-model is an exact rational conformal field theory. Compared to the principal model, it contains a Wess-Zumino term, which is topological and gives rise to a Dirac monopole quantization (put differently, the Kalb-Ramond three-form is classified according to the third homotopy group of the group manifold, which in the compact case is always $\pi_3(G) = \mathbb{Z}$). The radius is therefore quantized and this creates an obstruction when trying to let it vary continuously while keeping conformal invariance.

A possible way out would be to abandon conformal invariance. The perturbed sigma-model at hand is no longer an infra-red fixed point and various parameters (such as the radius of S^3 in the above example) now are running with the renormalization-group (RG) energy scale. It is tempting therefore, to interpret the RG scale as a cosmological time and the various (new) fixed points as steady or final states of the cosmological evolution. Such identification is however questionable because time is expected to appear as a genuine dimension of the target space of a critical string and nothing a priori guarantees that its evolution is the RG-flow of a non-critical string with one dimension less. Indeed, string backgrounds satisfy second-order equations while RG-flow equations are first-order. Hence, there is no reason to trust the above approach generically, although the dissipative nature of the dilaton may help in some circumstances to transform second-order into first-order equations. In some other examples, off criticality can generate a Liouville mode that may eventually play the role of the time direction, putting thereby on a firmer ground the cosmological interpretation of the RG evolution (see for instance refs. [1–5]). Despite these reservations, the method remains useful in order to acquire a wider perspective of the RG landscape around a given spatial string solution.¹

Instead of abandoning criticality, one can try to keep it or restore it. Starting from a conformal sigma-model with spatial target space, we add an extra time direction (which requires adjusting the internal manifold) and let some parameters explicitly depend on it. Time evolution of these parameters is dictated by the critical string equations of motion. This procedure may go under the name of *dynamical promotion of the RG-flow* although, depending on the parameter and the regime, this evolution may or may not be the one obtained in the absence of time, by following the RG-flow. An extreme situation correspond to the case where the parameter is a coupling constant associated with a truly marginal deformation of the original conformal sigma-model. Such a coupling does not run under the RG whereas it can acquire in general a non-trivial time dependence.

In the present note, we analyze both the RG-flow properties and its dynamical promotion in the following framework. We start with a conformal sigma-model with space-like d -dimensional target space. We assume the metric be Einstein with constant curvature, while criticality is achieved with a Kalb-Ramond field and no dilaton. The off-critical excursion is realized by switching on an overall scale factor c for the metric that does not affect the antisymmetric tensor. We perform the RG analysis and show, at lowest order in

¹A renormalization-group approach to cosmology has been put forward by other authors [6]. That perspective is however slightly different from ours.

α' , that no other fixed point exists away from $c = 1$.

We further investigate the case of the three-sphere. There, in a mini-superspace approximation, we consider a three-parameter perturbation written in terms of left-invariant currents describing a squashed three-sphere sustained by a Kalb-Ramond field. The RG-flow equations turn out to be equivalent to the Halphen system that we can explicitly solve in terms of Eisenstein series. The modular properties of the latter allow to demonstrate that, no matter the initial conditions, the system flows towards the round sphere. Again no other fixed point exists away from the $SU(2)$ WZW model.

It is remarkable that in all cases at hand — global scale factor or anisotropic perturbations, the RG-flow is in fact a *Ricci flow* driven by a connection with torsion, due to the Kalb-Rammond background. Ricci flows have attracted much attention recently [7], not only in mathematics but also in the physics literature, in relation to various issues of string theory such as tachyon condensation or time dependence (see e.g. [8–10]).

We investigate time-dependent solutions by considering a family of backgrounds expressed as the warped product of the original d -dimensional manifold with an extra time direction, while keeping the B field unperturbed.² The one-loop Weyl-invariance equations are satisfied for the full $(d + 1)$ -dimensional space-time provided a background dilaton is also switched on in the time direction. In the case of the three-sphere, we generalize the time warping by implementing time dependence in all three parameters that were studied in the RG approach — anisotropic warping.

The kind of solutions we obtain when only the overall scale factor is made time-dependent (simple warping) is by construction of the Friedmann-Robertson-Walker type. The system exhibits a friction behaviour. Whenever the latter dominates in the solution, we recover the RG-flow of the d -dimensional sigma-model studied previously. In this regime, the off-critical string generates a Liouville mode which becomes the “missing time” with a background charge that is identified with the friction coefficient. Warping coincides then with Liouville dressing. As a consequence, the dilaton of the full $(d + 1)$ -dimensional target space is linear in this regime.

In general, the solution turns out to describe a universe undergoing a power-law expansion (big-bang solution) superimposed with a damped oscillation governed by the dilaton and the axion. This kind of cosmologies was considered e.g. in [11, 12].

The case of more anisotropic warping that is studied for the sphere can be summarized as follows: the equations of motion are found to be a generalization of the Halphen system with a friction term. The solution exhibits the oscillating behaviour met previously. In particular, in the linearized limit, around the WZW point, all three parameters oscillate in phase and the solution remains qualitatively the same.

The paper is organized as follows. The RG-flow analysis is performed in section 2 first for the general case with perturbation of the overall factor and then for the three-sphere with more parameters for RG motion. In section 3 we introduce a time coordinate, in the manner described previously and take the various coupling constants perturbed so far

²This ensures that in the case of compact group manifolds, the quantization condition appearing in the WZW point is satisfied despite of the warping and is no longer an obstruction for finding new critical solutions.

to be time-dependent. This allows us to write down the equations of motion which are a generalization of the equations for the RG-flow and take the form of the motion of a point particle in presence of a friction term. The system is non linear and we were not able to find a closed solution, but both numerical and asymptotic analyzes are possible. They show that the universe we describe grows as a superposition of a big-bang expansion and a damped oscillation, and eventually converges to a linear dilaton solution [13, 14]. Section 4 is actually devoted to the proper cosmological analysis of the solution, performed by moving to the Einstein frame. A summary follows in section 5.

2. The renormalization-group-flow approach

2.1 The behaviour of the overall scale in general situations

We would like to study the RG-flow for the coupling of the metric in a system without dilaton, i.e. for the sigma model

$$\mathcal{L}(c) = \frac{1}{2} (cg_{\mu\nu} + B_{\mu\nu}) \partial X^\mu \bar{\partial} X^\nu, \tag{2.1}$$

where $g_{\mu\nu}$ is a d -dimensional Euclidean metric and c and λ are constants. The antisymmetric tensor is tuned so that for $c = 1$ the model is conformal³ and for simplicity we require $g_{\mu\nu}$ to be Einstein with constant curvature (that is always true e.g. for group manifolds, but this case is not exhaustive). Geometrically this means that $H = dB$ is the torsion that parallelizes the Riemannian metric $g_{\mu\nu}$, and no dilaton is required for conformal invariance.

Two-dimensional sigma models are perturbatively renormalizable. We will analyze the evolution of a certain class of deformations with special emphasis to their target-space interpretation, in the mini-superspace approximation. We will perform this analysis at one loop although it is in principle tractable at higher order.

Following [15–18] we define the connection

$$\Gamma^{-\mu}{}_{\nu\rho} = \left\{ \begin{matrix} \mu \\ \nu\rho \end{matrix} \right\} - \frac{1}{2} H^\mu{}_{\nu\rho} \tag{2.2}$$

and the corresponding Riemann tensor

$$R^{-\mu}{}_{\nu\rho\sigma} = \left(1 - \frac{1}{c^2} \right) R^\mu{}_{\nu\rho\sigma}. \tag{2.3}$$

In a dimensional-regularization scheme the one-loop counterterm reads:

$$\delta\mathcal{L}_{(1)} = \frac{\mu^\epsilon}{4\pi\epsilon} R_{\mu\nu}^- \partial X^\mu \bar{\partial} X^\nu \tag{2.4}$$

$$= \frac{\mu^\epsilon}{4\pi\epsilon} \frac{R}{d} \left(1 - \frac{1}{c^2} \right) g_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu, \tag{2.5}$$

where R is the Ricci scalar for $g_{\mu\nu}$. In particular, if we start with an $SU(2)$ WZW model at level k , we have $d = 3$ and $R = 6/k$.

³One should not confuse c with the central charge.

From the above expressions, we can determine the behaviour of c with respect to the scale μ , captured in the corresponding beta-function, $\beta(c) = dc/d\log\mu$. We find:

$$\beta(c) = \frac{R}{2\pi d} \left(1 - \frac{1}{c^2}\right), \tag{2.6}$$

which exhibits an infra-red fixed point at $c = 1$ that we already know. It is possible to go beyond one-loop and resum higher-order corrections. In the particular case of WZW models, where the target space is a group manifold, this amounts to a finite renormalization of the radius, or equivalently, a shift of the level of the affine algebra $k \rightarrow k + g^*$, where g^* is the dual Coxeter number of the algebra.

It is interesting to remark that given the form of the one-loop counterterm, the evolution equations take the form of a *Ricci flow* where the effect of the B field is taken into account by the connection Γ^- . In fact the equations above are equivalent to

$$\frac{dcg_{\mu\nu}}{d\log\mu} = \frac{1}{2\pi} R^-_{\mu\nu}. \tag{2.7}$$

As usual one should pay attention to the direction of the flow and it is hence useful to define an RG time variable $\tilde{t} = -\log\mu$ so that $g_{\mu\nu}(\tilde{t})$ describes the evolution of the system going towards the infra-red.

In order to compare the result with what we will find in the following, we introduce

$$\sigma(\tilde{t}) = \frac{1}{2} \log c(\tilde{t}). \tag{2.8}$$

Then the evolution of $\sigma(\tilde{t})$ going toward the infra-red gives:

$$\frac{d\sigma}{d\tilde{t}} = -\frac{R}{4\pi d} e^{-2\sigma(\tilde{t})} \left(1 - e^{-4\sigma(\tilde{t})}\right) \equiv -\frac{1}{4\pi} V'(\sigma(\tilde{t})) \tag{2.9}$$

with $V(\sigma)$ given in eq. (3.14). Equation (2.9) admits the implicit solution

$$\tilde{t} = -\frac{2\pi d}{R} \left(e^{2\sigma(\tilde{t})} - \arg \tanh \left(e^{2\sigma(\tilde{t})} \right) \right) + \text{cst.} \tag{2.10}$$

To summarize, in d dimensions and for constant-curvature metrics, the running of the one-parameter deformation that maintains the normalization for the topological term is a Ricci flow, and it does not produce any new non-trivial fixed points.

2.2 Generic perturbations of S^3 : the Halphen system

The case of WZW is of special interest. The underlying algebraic structure allows to switch on more general perturbations, while it remains possible to analyze their simultaneous evolution under the RG flow. Although the issue of metric perturbation has attracted much attention in the past (e.g. for $O(3)$ or $O(4)$ sigma models [19, 20]), we will here investigate a class of perturbations, for which the full analysis of the RG flow can also be carried out in the presence of a Kalb-Ramond field.

Let us for the moment consider a WZW model on a generic compact semi-simple group G . Its action is as in eq. (2.1) with $c = 1$. The corresponding algebra \mathfrak{g} has generators $\{T^\alpha\}$ and structure constants $f^{\alpha\beta}_\gamma$. The standard Killing form is⁴

$$ds^2 = \delta_{\alpha\beta} J^\alpha \otimes J^\beta, \tag{2.11}$$

where J^α are the left currents $J^\alpha = \sqrt{k} \operatorname{tr}(T^\alpha g^{-1} dg)$ for $g \in G$. We will consider a deformed WZW model with metric

$$ds^2 = g_{\alpha\beta} J^\alpha \otimes J^\beta = \sum_\alpha \gamma_\alpha(\mu) J^\alpha \otimes J^\alpha \tag{2.12}$$

(μ is the RG-scale) and the standard B field, which is fixed, being a topological term. It is convenient to go to the vielbein of the currents so that in particular the metric reads:

$$(g_{\alpha\beta}) = \begin{pmatrix} \gamma_1(\mu) & & & \\ & \gamma_2(\mu) & & \\ & & \ddots & \\ & & & \gamma_n(\mu) \end{pmatrix}, \tag{2.13}$$

where $n = \dim G$ and $\gamma_\alpha(\mu)$ arbitrary positive functions.

The RG-flow equations are here

$$\frac{dg_{\alpha\beta}}{d \log \mu} = \frac{1}{2\pi} \left(R_{\alpha\beta} - \frac{1}{4} H_{\alpha\beta}^2 \right) = \frac{1}{2\pi} R^-_{\alpha\beta}, \tag{2.14}$$

where $R_{\alpha\beta}$ are the components of the Ricci tensor, $H = dB$ and $H_{\alpha\beta}^2 = H_{\alpha\gamma\delta} H_\beta^{\gamma\delta}$. We observe that the RG-flow equations for the perturbation pattern at hand are again governed by a Ricci flow with torsion, as in section 2.1.

Although the full analysis is tractable for any compact group G , we will here focus on the $SU(2)$, where, as we will see below, the flow equations can be solved explicitly. The metric (2.13) has now only three entries: $\gamma_1(\mu), \gamma_2(\mu), \gamma_3(\mu)$. Using (A.9), we obtain the following Ricci tensor:

$$(R_{\alpha\beta}) = \frac{1}{2} \begin{pmatrix} \frac{\gamma_1^2 - (\gamma_2 - \gamma_3)^2}{\gamma_2 \gamma_3} & & \\ & \frac{\gamma_2^2 - (\gamma_1 - \gamma_3)^2}{\gamma_1 \gamma_3} & \\ & & \frac{\gamma_3^2 - (\gamma_1 - \gamma_2)^2}{\gamma_1 \gamma_2} \end{pmatrix}, \tag{2.15}$$

while the Kalb-Ramond term reads:

$$(H_{\alpha\beta}^2) = 2 \begin{pmatrix} \frac{1}{\gamma_2 \gamma_3} & & \\ & \frac{1}{\gamma_3 \gamma_1} & \\ & & \frac{1}{\gamma_1 \gamma_2} \end{pmatrix}. \tag{2.16}$$

⁴We will absorb the level k of the algebra in the currents. Therefore, in terms of the structure constants, $H = f_{\alpha\beta\gamma} J^\alpha \wedge J^\beta \wedge J^\gamma$, where the first indices in f are lowered with the Killing metric (H is the standard field for the unperturbed WZW model). Its square is contracted with the new metric though. Explicitly, $H_{\alpha\beta}^2 = f_{\alpha\gamma\delta} f_{\beta\gamma'\delta'} g^{\gamma\gamma'} g^{\delta\delta'}$.

We introduce as before an RG-time pointing towards the infra-red,

$$d\tilde{t} = -\frac{1}{2\pi\gamma_1(\mu)\gamma_2(\mu)\gamma_3(\mu)}d\log\mu, \tag{2.17}$$

where we have also reabsorbed the product of the three γ_α 's. These are positive functions and such a rescaling is therefore harmless since it does not spoil the monotonic evolution of \tilde{t} with μ (\tilde{t} increases from minus infinity to plus infinity when flowing to the infra-red from infinite to zero μ). Putting everything together one obtains the following RG equations:

$$\begin{cases} 2\frac{\dot{\gamma}_1}{\gamma_1} = (\gamma_2 - \gamma_3)^2 - \gamma_1^2 + 1, \\ 2\frac{\dot{\gamma}_2}{\gamma_2} = (\gamma_3 - \gamma_1)^2 - \gamma_2^2 + 1, \\ 2\frac{\dot{\gamma}_3}{\gamma_3} = (\gamma_1 - \gamma_2)^2 - \gamma_3^2 + 1, \end{cases} \tag{2.18}$$

where dot denotes the derivative with respect to \tilde{t} .

The evolution of Ricci flows on general homogeneous or locally homogeneous spaces has been studied from various perspectives (see e.g. [7, 21]). In the absence of torsion, the last constant term in (2.18) is missing and the flow converges towards the round sphere, of *vanishing* radius though. The presence of torsion does not alter this behaviour but affects the radius of the sphere which stabilizes to \sqrt{k} because all γ_α 's now converge to one.⁵ This non-trivial infra-red fixed point corresponds to the $SU(2)_k$ WZW model.

The above results on the convergence of the flow are based on asymptotic analysis. However, as already advertised, the Ricci-flow equations (2.18) can be solved explicitly in the case at hand. Indeed, setting

$$\tilde{t} = \log(T + T_0), \tag{2.19}$$

which amounts in identifying $T \in [0, +\infty[$ with a monotonic function of the inverse energy scale $1/\mu$, and

$$\Omega_1 = \frac{\gamma_2\gamma_3}{T + T_0}, \quad \Omega_2 = \frac{\gamma_3\gamma_1}{T + T_0}, \quad \Omega_3 = \frac{\gamma_1\gamma_2}{T + T_0}, \tag{2.20}$$

equations (2.18) are recast as:

$$\begin{cases} \dot{\Omega}_1 = \Omega_2\Omega_3 - \Omega_1(\Omega_2 + \Omega_3), \\ \dot{\Omega}_2 = \Omega_3\Omega_1 - \Omega_2(\Omega_3 + \Omega_1), \\ \dot{\Omega}_3 = \Omega_1\Omega_2 - \Omega_3(\Omega_1 + \Omega_2), \end{cases} \tag{2.21}$$

where the dot now denotes the derivative with respect to the new time⁶ T . The arbitrary constant T_0 does not play any significant role.

⁵Note that the differential systems under consideration are parabolic, hence not invariant under time reversal.

⁶In the case of a system without torsion one would obtain the same equations by defining $\Omega_1 = \gamma_2\gamma_3$ and cyclic and keeping the same time variable \tilde{t} .

This is the celebrated Halphen system that was studied in the 19th century, also called Darboux-Halphen because the equations were written by Darboux on the analysis of triply orthogonal surfaces [22] and solved three years later by Halphen [23, 24]. It has appeared since then in several instances in physics, as e.g. in the search for self-dual or anti-self-dual reductions in four-dimensional Euclidean gravity (Bianchi IX class with SU(2) isometry) [25] or in the scattering of SU(2)-Yang-Mills monopoles [26]. The function $Y = -2(\Omega_1 + \Omega_2 + \Omega_3)$ satisfies

$$\dot{Y} = 2Y \ddot{Y} - 3\dot{Y}^2, \tag{2.22}$$

known as the Chazy equation [27, 28]. Note that integrable ordinary differential equations like Chazy's or Halphen's turn out to be systematically related to self-dual Yang-Mills reductions.

The Halphen and the Chazy equations possess remarkable properties. If $\omega_\alpha(z)$ provide a solution to the Halphen system for generic $z \in \mathbb{C}$ (the dot stands then for d/dz), then so do

$$\tilde{\omega}_\alpha(z) = \frac{1}{(cz+d)^2} \omega_\alpha\left(\frac{az+b}{cz+d}\right) + \frac{c}{cz+d}, \tag{2.23}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C})$. Similarly, if $y(z)$ is a solution of the Chazy equation, so is

$$\tilde{y}(z) = \frac{1}{(cz+d)^2} y\left(\frac{az+b}{cz+d}\right) - \frac{6c}{cz+d}. \tag{2.24}$$

A class of solutions are expressed as [29]

$$\omega_\alpha(z) = -\frac{1}{2} \frac{d}{dz} \log E_\alpha(z), \tag{2.25}$$

where E_α form a triplet of modular forms of weight two for $\Gamma(2) \subset PSL(2, \mathbb{Z})$ (see e.g. [30, 31]), transforming thereby as

$$z \rightarrow -\frac{1}{z} : \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \rightarrow z^2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \tag{2.26}$$

and

$$z \rightarrow z+1 : \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \rightarrow - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}. \tag{2.27}$$

The class of solutions under consideration *does not* capture situations where some ω_α 's are equal. In our physical set-up, this would correspond to deformations of the three-sphere that preserve some extra isometries, whereas all-different ω_α 's restrict the isometry group to SU(2), strictly. We will discuss this possibility towards the end of the present section, but the following should already be stressed: eqs. (2.21) imply that if $\Omega_\alpha = \Omega_\beta$ then $\dot{\Omega}_\alpha = \dot{\Omega}_\beta$ and therefore Ω_α and Ω_β never cross unless they are equal at any time. For the moment, we will focus on the situation where $\Omega_1 \neq \Omega_2 \neq \Omega_3$. We can express the E_α as

$$E_1 = \frac{d\lambda/dz}{\lambda}, \quad E_2 = \frac{d\lambda/dz}{\lambda-1}, \quad E_3 = \frac{d\lambda/dz}{\lambda(\lambda-1)}, \tag{2.28}$$

and recast the Halphen system in terms of the Schwartz equation for λ :

$$\frac{d^3\lambda/dz^3}{d\lambda/dz} - \frac{3}{2} \left(\frac{d^2\lambda/dz^2}{d\lambda/dz} \right)^2 = -\frac{1}{2} \left(\frac{1}{\lambda^2} + \frac{1}{(\lambda-1)^2} - \frac{1}{\lambda(\lambda-1)} \right) \left(\frac{d\lambda}{dz} \right)^2. \quad (2.29)$$

The elliptic modular function

$$\lambda = \frac{\vartheta_2^4}{\vartheta_3^4} \quad (2.30)$$

is a celebrated solution for this equation (see appendix B for a reminder on Jacobi theta functions). This provides a particular solution of the Halphen system in terms of the three $\Gamma(2)$ weight-two *Eisenstein series*. Correspondingly,

$$y = 12 \frac{d}{dz} \log \eta, \quad (2.31)$$

where η is the Dedekind function.

For our present purposes, we must focus on real solutions Ω_α for real time T ($z = iT$). These are obtained as

$$\Omega_\alpha(T) = i\omega_\alpha(iT) = -\frac{1}{2} \frac{d}{dT} \log E_\alpha(iT) \quad (2.32)$$

and

$$Y(T) = iy(iT) = \frac{d}{dT} \log E_1 E_2 E_3. \quad (2.33)$$

The modular properties of the functions under consideration set stringent constraints between the asymptotics of the solution and its initial conditions: large- T and small- T regimes are related by $T \leftrightarrow 1/T$. Assuming $\Omega_\alpha^0 \equiv \Omega_\alpha(0)$ finite and using (2.26) and (2.32), the asymptotic behaviour ($T \rightarrow \infty$) is found to be *universally*

$$\Omega_\alpha = \frac{1}{T} + \text{subleading}. \quad (2.34)$$

This provides an elegant proof of the universality of generic SU(2) Ricci flows in the presence of torsion towards the corresponding WZW infra-red fixed point.

One can further assume Ω_α^0 finite and positive. This is natural for describing the initial deformation of a three-sphere and sufficient to show that *all* Ω_α remain positive at any later time. Indeed, suppose that $0 < \Omega_1^0 < \Omega_2^0 < \Omega_3^0$ and that Ω_1 has reached at time t_1 the value $\Omega_1^1 = 0$, while $\Omega_2^1, \Omega_3^1 > 0$. From eqs. (2.21) we conclude that at time t_1 , $\dot{\Omega}_2^1 = \dot{\Omega}_3^1 = -\Omega_2^1 \Omega_3^1 < 0$ and $\dot{\Omega}_1^1 = \Omega_2^1 \Omega_3^1 > 0$. This latter inequality implies that Ω_1 vanishes at t_1 while it is increasing, passing therefore from negative to positive values. This could only happen if Ω_1^0 were negative, which contradicts the original assumption. However, if indeed $\Omega_1^0 < 0$ and $\Omega_2^0, \Omega_3^0 > 0$, there is a time t_1 where Ω_1 becomes positive and remains positive together with Ω_2 and Ω_3 until they reach the asymptotic region, where they all satisfy (2.34). The generic behaviour of a positive-initial-value solution is given in figure 1.

Although the Halphen system can be solved, its deep nature makes it difficult to establish the correspondence between a given set of initial conditions and the modular forms E_α (see eqs. (2.25) or (2.32)) necessary to provide the actual solution. Moreover, it

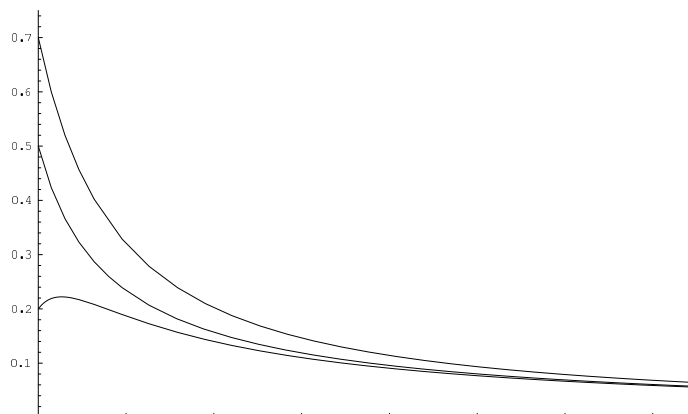


Figure 1: A generic solution of the Halphen system for positive $\Omega_1^0 < \Omega_2^0 < \Omega_3^0$.

is important to stress that solutions exist, for which the initial conditions Ω_α^0 are not all positive and not even finite. Hence, for these solutions, (2.34) does not hold. This happens e.g. for the particular solution given in eqs. (2.28) and (2.30), where $T = 0$ is a simple pole with positive (unit) residue for Ω_2, Ω_3 and a double pole with negative residue $(-\pi/2)$ for Ω_1 . As a consequence, Ω_1 is negative and increases, at large T , exponentially towards zero, while Ω_3 is positive and decreases exponentially towards zero; Ω_2 is positive and decreases exponentially towards $\pi/2$. Solutions with negative Ω 's were considered in refs. [26, 32] for the description of the configuration manifold of two $SU(2)$ monopoles.

The existence of poles is generic for *all* solutions of the Halphen system. For solutions corresponding to a set of finite and different initial values, these poles are pushed behind $T = 0$. This is related to the following general property: Halphen's solutions possess a natural movable boundary [29]. They exist in a domain of \mathbb{C} , where they are holomorphic and single valued, and this domain has a boundary that contains a dense set of essential singularities. The precise location of this boundary depends on the initial conditions.

We would like finally to mention that the Halphen system admits also solutions which are *not* based on elliptic functions. These solutions have power-like dependence on time:

$$\begin{cases} \Omega_1 = \frac{1}{T+A} + \frac{C}{(T+A)^2}, \\ \Omega_2 = \Omega_3 = \frac{1}{T+A}. \end{cases} \quad (2.35)$$

They are actually the *most general* solutions with $\Omega_2 = \Omega_3$ and describe axisymmetric deformations of the three-sphere,⁷ namely deformations preserving an $SU(2) \times U(1)$. The constants (A, C) are arbitrary and determined by the initial conditions. This class is closed under $PSL(2, \mathbb{C})$ transformations. Indeed, using (2.23), we learn that under $T \rightarrow 1/T$, $(A, C) \rightarrow (1/A, -C/A^2)$, while for $T \rightarrow T+1$, $(A, C) \rightarrow (A+1, C)$. Regularity requires $\Omega_{2,3}^0 \equiv 1/A > 0$, which ensures that the pole of Ω_α is located at $T < 0$. Furthermore, the asymptotic behaviour is again universal as in eq. (2.34). No new fixed point appears therefore in this

⁷Notice that the equations at hand being first order, setting $\Omega_2^0 = \Omega_3^0$ guarantees that $\Omega_2 = \Omega_3$ at any subsequent time.

case either. The RG flow of perturbations of this type, namely with $\gamma_1 \neq \gamma_2 = \gamma_3$, was analyzed in [20], without torsion though. When translated into the language of self-dual four-dimensional Euclidean metrics with $SU(2)$ isometry, solutions (2.35) correspond to the general Taub-NUT family, including Eguchi-Hanson metrics [25].

The case of $\Omega_1 = \Omega_2 = \Omega_3$ corresponds to fully isotropic deformations studied in the previous section. In this case, the solution is unique: $\Omega_\alpha = 1/T+A$.

To summarize and conclude the present analysis, solutions of the Halphen system fall in three classes: $\Omega_1 \neq \Omega_2 \neq \Omega_3$, $\Omega_1 \neq \Omega_2 = \Omega_3$ and $\Omega_1 = \Omega_2 = \Omega_3$. The corresponding three-manifolds, target spaces of the sigma-model, are homogeneous with isometry group $SU(2)$, $SU(2) \times U(1)$ and $SU(2) \times SU(2)$, respectively. This captures all possible isometry groups for a general three-sphere, the latter, most symmetric case corresponding to the usual round sphere. The Ricci flow describing the renormalization of the sigma-model leads unavoidably to the round sphere, which is therefore the unique perturbative infra-red fixed point, found at one loop.

3. Space-time solutions

3.1 Equations of motion for the dilaton and the scale factor

As advertised in the introduction, our aim is now to describe a generalization of the previous construction obtained by introducing an extra time dimension and treating the coupling as a time-dependent field. This clearly bears many resemblances with the usual Liouville dressing of [1, 33–35]. In other words, we would like to write down the Weyl-invariance equations for the following sigma model:

$$S = \frac{1}{2} \int d^2z \left[-\partial t \bar{\partial} t + (c(t)g_{\mu\nu} + B_{\mu\nu}) \partial X^\mu \bar{\partial} X^\nu + R\Phi(t) \right], \quad (3.1)$$

where, g and B are background fields solving the d -dimensional equations of motion. Modulo the time-dependent dilaton, the system in eq. (2.1) appears thus as a description of a constant-time slice.

Let us start by rewriting the $d+1$ dimensional metric in the form of a Weyl rescaling,

$$\bar{g}_{MN} = e^{2\sigma(t)} \begin{pmatrix} -e^{-2\sigma(t)} & 0 \\ 0 & g_{\mu\nu} \end{pmatrix} = e^{2\sigma(t)} g_{MN}, \quad (3.2)$$

where $c(t) = e^{2\sigma(t)}$, as before. This means in particular that the Ricci tensor can be written as

$$\bar{R}_{MN} = R_{MN} - g_{MN} K_L^L - (d-1) K_{MN}, \quad (3.3)$$

where K_{MN} is defined as

$$K_M^N = -\partial_M \sigma g^{NL} \partial_L \sigma + g^{NL} (\partial_M \partial_L \sigma - \Gamma_{ML}^P \partial_P \sigma) + \frac{1}{2} g^{LP} \partial_L \sigma \partial_P \sigma \delta_M^N, \quad (3.4)$$

$$K_{MN} = g_{NL} K_M^L. \quad (3.5)$$

After some algebra one finds

$$\bar{R}_{tt} = -d(\ddot{\sigma}(t) + \dot{\sigma}^2(t)), \quad (3.6a)$$

$$\bar{R}_{t\mu} = 0, \quad (3.6b)$$

$$\bar{R}_{\mu\nu} = R_{\mu\nu} + \bar{g}_{\mu\nu}(d\dot{\sigma}^2(t) + \ddot{\sigma}(t)). \quad (3.6c)$$

The $(d + 1)$ -dimensional metric is not Einstein, so we expect the presence of a time-dependent dilaton. The equations of motion read:

$$\begin{cases} \beta_\Phi = -\frac{1}{2}\bar{\square}\Phi + \bar{\nabla}_M\Phi\bar{\nabla}^M\Phi - \frac{1}{24}H_{MNP}\bar{H}^{MNP} = \frac{d+c_1-25}{6}, \\ \beta_G = \bar{R}_{MN} - \frac{1}{4}\bar{H}_{MN}^2 + 2\bar{\nabla}_M\bar{\nabla}_N\Phi = 0, \\ \beta_B = -\frac{1}{2}\bar{\nabla}^M H_{MNP} + \partial^M\Phi H_{MNP} = 0. \end{cases} \quad (3.7)$$

Several remarks are in order here.

- As already pointed out, the action in eq. (3.1) does not describe by itself a critical string theory. An internal conformal field theory of central charge c_1 must be superimposed to the above sigma-model. Requiring that we are in the perturbative regime with respect to α' , we expect: (i) the model in eq. (3.1) to have a central charge of the order of $d + 1$ and (ii) the internal CFT with $c_1 \approx 25 - d$ to have a geometrical interpretation with a $(25 - d)$ -dimensional compact target space (we refer here to the bosonic string).
- If the scale factor were chosen to be kept constant and equal to its critical value $c = 1$, the natural solution to the above equations would be the *linear dilaton* $\Phi = \pm qt$. This solution is actually an exact CFT with central charge $1 - 3q^2 < 1$ (the minus sign is due to the time-like direction). The central charge c_d of the d -dimensional sigma model (decoupled in this case) depends on the geometry (the curvature) of the target space. For some spherical-like geometry, one expects $c_d \lesssim d$ and therefore the internal CFT must have $c_1 \gtrsim 25 - d$. This imposes a locally hyperbolic internal target space that can be e.g. a quotient of a hyperbolic plane H_{25-d} by some discrete subgroup.

The $\beta_\Phi = \text{const.}$ equation can be seen as a consequence of the Bianchi identities and the $\beta_B = 0$ and $\beta_G = 0$ equations. Explicitly it reads:

$$\beta_\Phi = -\frac{1}{2}\ddot{\Phi} + \dot{\Phi}\left(\dot{\Phi} - \frac{d}{2}\dot{\sigma}\right) - \frac{1}{6}e^{-6\sigma}R = \frac{d + c_1 - 25}{6} \quad (3.8)$$

The $\beta_B = 0$ condition is satisfied by our choice of Kalb-Ramond field (see section 2.1) and a time-dependent dilaton $\Phi = \Phi(t)$. So, let us consider the remaining $\beta_G = 0$ equation. First of all we separate the time component (all terms retain a block-matrix structure):

$$\begin{cases} R_{tt} + 2\partial_t\dot{\Phi}(t) = -d(\ddot{\sigma}(t) + \dot{\sigma}^2(t)) + 2\ddot{\Phi}(t) = 0, \\ R_{\mu\nu}(1 - e^{-4\sigma(t)}) + \bar{g}_{\mu\nu}(d\dot{\sigma}^2(t) + \ddot{\sigma}(t) - 2\dot{\sigma}(t)\dot{\Phi}(t)) = 0, \end{cases} \quad (3.9)$$

where we used the equations of motion for the $\sigma = 0$ conformal system

$$R_{\mu\nu} = \frac{1}{4}H_{\mu\nu}^2. \quad (3.10)$$

Taking the trace with \bar{g}^{MN} one obtains the system:

$$\begin{cases} d(\ddot{\sigma}(t) + \dot{\sigma}^2(t)) - 2\ddot{\Phi}(t) = 0, \\ Re^{-2\sigma(t)}(1 - e^{-4\sigma(t)}) + d(d\dot{\sigma}^2(t) + \ddot{\sigma}(t) - 2\dot{\sigma}(t)\dot{\Phi}(t)) = 0. \end{cases} \quad (3.11)$$

Introducing

$$Q(t) = -\dot{\Phi}(t) + \frac{d}{2}\dot{\sigma}(t), \quad (3.12)$$

the equations become:

$$\begin{cases} \dot{Q}(t) = -\frac{d}{2}\dot{\sigma}^2(t), \\ \ddot{\sigma}(t) = -\frac{R}{d}e^{-2\sigma(t)}(1 - e^{-4\sigma(t)}) - 2\dot{\sigma}(t)Q(t). \end{cases} \quad (3.13)$$

The second equation has precisely the structure of the equation of motion of a point particle in a potential

$$V(\sigma) = \frac{R}{6d}e^{-6\sigma}(1 - 3e^{4\sigma}) + \text{cst.}, \quad (3.14)$$

in the presence of a time-dependent friction coefficient $Q(t)$. In the $Q \rightarrow \infty$ limit we recover the same equation as in eq. (2.9) with the same potential $V(\sigma)$ when identifying the *energy scale* μ for the running, off-shell (i.e. non-conformal), d -dimensional system with the *time direction* according to

$$\log \mu = -\frac{2\pi t}{Q}. \quad (3.15)$$

This was to be expected since what we find is the usual link between the RG-flow equations and the conformal condition typical of a Liouville dressing (see *e.g.* [1, 2, 33, 35]). This identification holds, however, only in the large- Q regime.

A useful reformulation of the equations is obtained by multiplying the second equation by $\dot{\sigma}$ and introducing the energy $E = \dot{\sigma}^2/2$:

$$\begin{cases} \dot{Q}(t) = -dE(t), \\ E(t) = -V(\sigma(t)) + \frac{2}{d}Q^2(t), \end{cases} \quad (3.16)$$

which in turn allows us to write down an equation for Q as a function of σ :

$$\frac{d}{d\sigma}Q(\sigma) = -d\sqrt{\frac{Q(\sigma)^2}{d} - \frac{V(\sigma)}{2}}. \quad (3.17)$$

3.2 Linearization

The system (3.13) can be solved numerically and typical results for large $Q(0)$ and small $Q(0)$ are shown in figure 2. Since we describe a dissipative motion, the system relaxes to equilibrium with or without fluctuations.

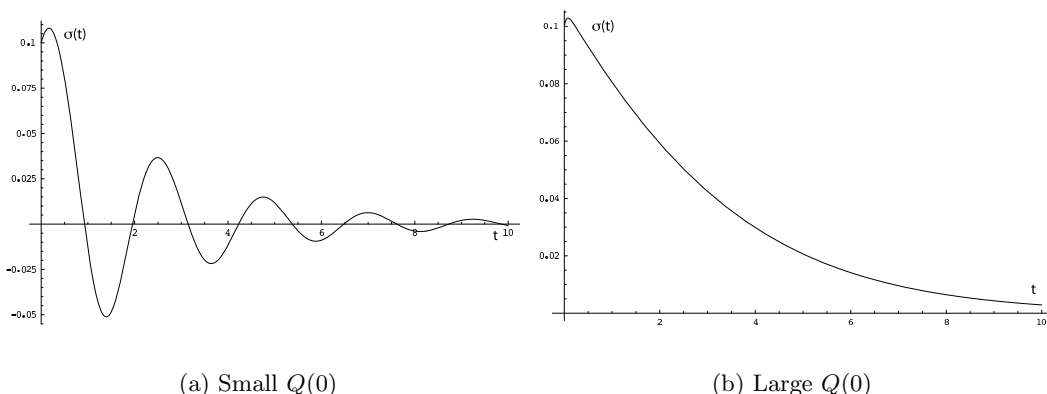


Figure 2: Typical behaviour for $\sigma(t)$ in the non-linear system. For (a) small and (b) large (positive) initial values of $Q(t)$ (numerical integration with (a) $(Q(0), \sigma(0), \dot{\sigma}(0)) = (0.5, 0.1, 0.1)$) and (b) $(Q(0), \sigma(0), \dot{\sigma}(0)) = (10, 0.1, 0.1)$).

The solutions we found are in principle only valid at first order in α' so one may wonder about their stability. Actually, it turns out that if the spatial part is supersymmetric (which is the case if the model in eq. (2.1) is of the Wess-Zumino-Witten type), then so is the spacetime in eq. (3.2). This is immediate once one notices that the Weyl tensor for \bar{g} is the same as for the Cartesian product with line element $ds^2 = -dt^2 + g_{\mu\nu}dx^\mu dx^\nu$ and therefore vanishes identically.

Around the conformal point. A further step can be made via linearization. To study the behaviour in a neighbourhood of the $\sigma = 0$ conformal point we introduce

$$\Sigma(t) = \dot{\sigma}(t) \tag{3.18}$$

and derive the first-order system

$$\begin{cases} \dot{Q}(t) = -\frac{d}{2}\Sigma^2(t), \\ \dot{\sigma}(t) = \Sigma(t), \\ \dot{\Sigma}(t) = -V'(\sigma(t)) - 2\Sigma(t)Q(t), \end{cases} \tag{3.19}$$

which has a fixed point for $(Q(t), \sigma(t), \Sigma(t)) = (\bar{Q}, 0, 0)$, \bar{Q} being a constant. Around this point the asymptotic behaviour is described by

$$\begin{cases} \dot{Q}(t) = 0, \\ \dot{\sigma}(t) = \Sigma(t), \\ \dot{\Sigma}(t) = -V''(0)\sigma(t) - 2\bar{Q}\Sigma(t) = -\frac{4R}{d}\sigma(t) - 2\bar{Q}\Sigma(t). \end{cases} \tag{3.20}$$

Q decouples (and remains constant) and the only non-trivial remaining equation of motion is

$$\frac{d^2\sigma(t)}{dt^2} = -\frac{4R}{d}\sigma(t) - 2\bar{Q}\frac{d\sigma(t)}{dt}, \tag{3.21}$$

which can be integrated to give

$$\sigma(t) = C_1 \exp \left[- \left(\bar{Q} + \sqrt{\bar{Q}^2 - \frac{4R}{d}} \right) t \right] + C_2 \exp \left[- \left(\bar{Q} - \sqrt{\bar{Q}^2 - \frac{4R}{d}} \right) t \right], \quad (3.22a)$$

$$\Phi(t) = \Phi_0 + \frac{dC_1}{2} \exp \left[- \left(\bar{Q} + \sqrt{\bar{Q}^2 - \frac{4R}{d}} \right) t \right] + \frac{dC_2}{2} \exp \left[- \left(\bar{Q} - \sqrt{\bar{Q}^2 - \frac{4R}{d}} \right) t \right] - \bar{Q}t, \quad (3.22b)$$

where C_1 and C_2 are integration constants.

For positive \bar{Q} the solution converges to $\sigma = 0$ with or without oscillations depending on whether $\bar{Q}^2 \leq 4R/d$, which clarifies the meaning of “large” and “small” $Q(0)$ in figure 2.

In terms of $\sigma(t)$ and $\Phi(t)$, this limit solution is

$$\sigma(t) \xrightarrow[t \rightarrow \infty]{} 0 \qquad \Phi(t) \sim \Phi_0 - \bar{Q}t, \quad (3.23)$$

which is the conformal model in eq. (2.1) plus a linear dilaton.

Large σ . In the large- σ limit, the solution goes to flat space. To study its behaviour let us consider the system

$$\begin{cases} \dot{Q}(t) = -d \left(-V(\sigma) + \frac{2}{d} Q(t)^2 \right), \\ \ddot{\sigma}(t) = -V'(\sigma(t)) - 2\dot{\sigma}(t)Q(t). \end{cases} \quad (3.24)$$

In the large- σ limit $V(\sigma) \rightarrow C$ and $V'(\sigma) \rightarrow 0$ so it reduces to

$$\begin{cases} \dot{Q}(t) = -d \left(-C + \frac{2}{d} Q(t)^2 \right), \\ \ddot{\sigma}(t) = -2\dot{\sigma}(t)Q(t), \end{cases} \quad (3.25)$$

which can be solved analytically:

$$Q(t) = \sqrt{\frac{Cd}{2}} \tanh \left(2\sqrt{Cd}t + C_1 \right), \quad (3.26)$$

$$\sigma(t) = C_3 + C_2 \arctan \left(\sinh C_1 + \cosh C_1 \tanh \left(\sqrt{\frac{Cd}{2}} t \right) \right) \quad (3.27)$$

$$\Phi(t) = \Phi_0 - \frac{1}{2\sqrt{2}} \log \left(\cosh \left(2\sqrt{Cd}t + C_1 \right) \right) + \frac{d}{2} \sigma(t) \quad (3.28)$$

where C_1, C_2, C_3 are integration constants. As expected, for large t , both $Q(t)$ and $\sigma(t)$ asymptotically converge to constant values and the background is flat with a linear dilaton.

3.3 The meaning of \bar{Q}

The parameter \bar{Q} is interpreted as the background charge that drives the dilaton, in the regime where the latter is linear with time. This parameter dictates the behaviour of the solution around the fixed point in many ways. Apart from discriminating between exponentially decreasing and oscillating solutions, it also measures the consistency of the perturbative string approach. In fact positive values of \bar{Q} correspond to a dilaton evolving

towards large negative values, i.e. towards the perturbative regime. On the other hand, negative values of \bar{Q} would lead to diverging solutions in which we would lose control over the approximations we made. It is worth to remark that in the non-linear dynamics $Q(t)$ is not allowed to change its sign if we make the hypothesis of unicity of the solution. In fact changing sign would require to cross the $Q = 0, \sigma = \text{const.}$ line corresponding to the Euclidean conformal model.

A final remark regards the consistency of the approximation for the dynamics one obtains from the RG-flow equation (2.9), corresponding to a $Q \rightarrow \infty$ limit. The linearized system (3.20) provides a justification for such limit: in fact the time scale for $Q(t)$ is comparably larger than the time scale for $\sigma(t)$ — to the point that the former decouples around the fixed point. For this reason it can be taken as a constant (fixed by the initial conditions) if we just concentrate on the evolution of the warp factor $\sigma(t)$.

3.4 The anisotropic deformations of the three-sphere

We will now study the anisotropic deformation of the SU(2) model (see section 2.2) when an extra time dimension is added to the system while the parameters γ_ν (eq. (2.13)) and the dilaton Φ are allowed to depend on it. Explicitly, the full four-dimensional, time-dependent metric under consideration is

$$\begin{aligned}
 ds^2 = & -dt^2 + \sum_{\nu=1}^3 \gamma_\nu(t) J^\nu J^\nu = -dt^2 + k \left((\gamma_2(t) \cos^2 \phi + \gamma_1(t) \sin^2 \phi) d\theta^2 + \right. \\
 & + (\gamma_1(t) \cos^2 \theta \cos^2 \phi + \gamma_3(t) \sin^2 \theta + \gamma_2(t) \cos^2 \theta \sin^2 \phi) d\psi^2 + \\
 & \left. + \gamma_3(t) d\phi^2 + 2\gamma_3(t) \sin \theta d\phi d\psi + 2(\gamma_1(t) - \gamma_2(t)) \cos \theta \cos \phi \sin \phi d\theta d\psi \right). \quad (3.29)
 \end{aligned}$$

We introduce again the functions $\sigma_\nu(t)$ as

$$\gamma_\nu(t) = e^{2\sigma_\nu(t)}, \quad (3.30)$$

and find the following equations of motion:

$$\begin{cases}
 2\ddot{\Phi}(t) = \sum_{\nu=1}^3 (\ddot{\sigma}_\nu(t) + \sigma_\nu(t)^2), \\
 \ddot{\sigma}_1(t) = \frac{R}{3} \left(1 - e^{4\sigma_1(t)} + (e^{2\sigma_2(t)} - e^{2\sigma_3(t)})^2 \right) \prod_{\nu=1}^3 e^{-2\sigma_\nu(t)} - \dot{\sigma}_1(t) \left(-2\dot{\Phi}(t) + \sum_{\nu=1}^3 \sigma_\nu(t) \right),
 \end{cases} \quad (3.31)$$

plus permutations, where $R = 6/k$ is the curvature of the undeformed three-sphere. Here, it is natural to define

$$Q(t) = -\dot{\Phi}(t) + \frac{1}{2} \sum_{\nu=1}^3 \dot{\sigma}_\nu(t) \quad (3.32)$$

and recast the system in the form

$$\begin{cases}
 \dot{Q}(t) = -\frac{1}{2} \sum_{\nu=1}^3 \dot{\sigma}_\nu^2(t), \\
 \ddot{\sigma}_1(t) = \frac{R}{3} \left(1 - e^{4\sigma_1(t)} + (e^{2\sigma_2(t)} - e^{2\sigma_3(t)})^2 \right) \prod_{\nu=1}^3 e^{-2\sigma_\nu(t)} - 2\dot{\sigma}_1(t)Q(t),
 \end{cases} \quad (3.33)$$

plus permutations.

The analysis of eqs. (3.33) can be performed in analogy with the system we obtained above, when considering only the overall breathing mode. We first observe that given the RG-flow equations⁸

$$\frac{d\sigma_\nu(\mu)}{d \log \mu} = \beta_\nu(\sigma(\mu)), \tag{3.34}$$

we obtain those for the time-dependent system as

$$\frac{d^2\sigma_\nu(t)}{dt^2} = \beta_\nu(\sigma(t)) - 2Q(t)\frac{d\sigma_\nu(t)}{dt} \tag{3.35}$$

where $Q(t)$ is the charge that naturally generalizes the one we found before. From these we conclude that the structure is again that of a dissipative system with the time-dependent friction coefficient given by the charge. We can further proceed by linearization around the $SU(2)$ wzw fixed point. This goes along the same lines as in the isotropic case and gives

$$\begin{cases} \dot{Q}(t) = 0, \\ \ddot{\sigma}_\nu(t) = -\frac{4R}{d}\sigma_\nu(t) - 2\bar{Q}\dot{\sigma}_\nu(t), \quad \nu = 1, 2, 3. \end{cases} \tag{3.36}$$

The charge $Q(t)$ decouples and remains constant $Q(t) = Q(0) = \bar{Q}$ while the three modes become independent from each other and obey the same differential equation. This means that they oscillate in phase with each other and eventually converge to the fixed point corresponding to $SU(2)_k$ wzw plus a linear dilaton.

All the considerations we made for the breathing mode then extend directly to this anisotropic generalization. The net effect of the time dependence is therefore to wash out the anisotropy, as was expected from the pure RG-flow analysis of section 2.2.

4. Cosmological interpretation

The type of backgrounds we are studying are time-dependent and as such can be of cosmological interest. For this reason, since there is a non-trivial dilaton, one should move to the Einstein frame (as opposed to the string frame we have been using thus far). The metric takes the form

$$\tilde{g}_{MN} = e^{-\Phi(t)/2}\bar{g}_{MN}, \tag{4.1}$$

and after a coordinate change

$$\tau(t) = \int_0^t e^{-\Phi(t')/4} dt', \tag{4.2}$$

it reduces to the same warped product form as in eq. (3.2):

$$\widetilde{ds}^2 = \tilde{g}_{MN} dx^M dx^N = -d\tau^2 + e^{2\sigma(t)-\Phi(t)/2} \Big|_{t=t(\tau)} g_{\mu\nu} dx^\mu dx^\nu = -d\tau^2 + a(\tau)g_{\mu\nu} dx^\mu dx^\nu \tag{4.3}$$

⁸Notice that $\beta_\nu(\sigma) = \beta_\nu(\gamma)/2\gamma_\nu$.

which has Hubble parameter:

$$H(\tau) = \frac{d}{d\tau} \log(a(\tau)) = e^{\Phi(t)/4} \left(2\dot{\sigma}(t) - \frac{\dot{\Phi}(t)}{2} \right) \Big|_{t=t(\tau)}. \quad (4.4)$$

Cosmologically interesting solutions are obtained when $d = 3$. In this case the Kalb-Ramond field $H_{[3]}$ is proportional to the volume form on g . This implies that $H_{\mu\nu}^2 \propto g_{\mu\nu}$ and then eq. (3.10) reduces to

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu}, \quad (4.5)$$

i.e. $g_{\mu\nu}$ is the metric of an Einstein three-manifold. The simplest case is a three-sphere for which we get a typical example of FRW space-time of the kind studied in [36–39]. As such it describes the time evolution of a homogeneous and isotropic⁹ space-time, or more generally of a space-time with the target-space symmetries of the conformal theory in eq. (2.1).

Some intuition about the time evolution can be developed if we take the linearized system in eq. (3.20).

Linear Dilaton. In fact, as remarked above, if we consider the large t limit, the solution asymptotically approaches a linear dilaton background (which was already studied from this point of view in [14]):

$$\sigma(t) \xrightarrow{t \rightarrow \infty} 0 \qquad Q(t) = \bar{Q} \qquad \Phi(t) \sim -\bar{Q}t, \quad (4.6)$$

hence one can easily verify that the metric in the Einstein frame is asymptotically

$$\widetilde{ds^2} \sim -d\tau^2 + \bar{Q}^2 \tau^2 g_{\mu\nu} dx^\mu dx^\nu, \quad (4.7)$$

which corresponds to an expanding universe with Hubble parameter

$$H(\tau) = \frac{2}{\tau} \quad (4.8)$$

and curvature

$$\tilde{R} \sim \frac{R + \bar{Q}^2 d(d-1)}{\bar{Q}^2 \tau^2}. \quad (4.9)$$

Exponentially decreasing σ . A similar result with a polynomial expansion is found if we consider an exponential decrease for $\sigma(t)$, or better for $c(t)$ (in the linear limit $c(t) - 1$ obeys the same equations as $\sigma(t)$) i.e. if we are in the large- \bar{Q} regime. After a redefinition of the variables we can write:

$$c(t) = e^{-\alpha t} + 1. \quad (4.10)$$

It is easy to check that in general¹⁰

$$\tau(t) = \int c(t)^{-d/16} e^{1/4 \int Q(t') dt'} dt, \quad (4.11)$$

⁹We will discuss in this section the isotropic situation only.

¹⁰Since $c(t) > 0$ by construction, the relation $\tau = \tau(t)$ is always invertible.

and in this linearized approximation this becomes

$$\tau(t) = \int (e^{-t} + 1)^{-d/16} e^{\bar{Q}t/4} dt. \tag{4.12}$$

Although this integral can be solved analytically:

$$\tau(u) = \frac{16}{d + 4\bar{Q}} \left(1 + \frac{1}{u}\right)^{-d/16} u^{\bar{Q}/4} (1 + u)^{d/16} {}_2F_1\left(\frac{d}{16}, \frac{d + 4\bar{Q}}{16}; \frac{d + 4\bar{Q}}{16} + 1, -u\right), \tag{4.13}$$

where $u = e^t$ and ${}_2F_1$ is an hypergeometric function.¹¹ It is better to consider the asymptotic behaviour. For $u \rightarrow \infty$ one finds that $\tau(u)$ and the warp factor $a(u)$ go as:

$$\tau(u) \underset{\tau \rightarrow \infty}{\sim} \frac{4}{\bar{Q}} u^{\bar{Q}/4}, \quad a(u) \underset{\tau \rightarrow \infty}{\sim} u^{\bar{Q}/2}, \tag{4.16}$$

and consistently with the results above for the linear dilaton case (which is precisely the large- u limit):

$$a(\tau) \underset{\tau \rightarrow \infty}{\sim} \tau^2; \tag{4.17}$$

similarly for small u :

$$\tau(u) \underset{\tau \rightarrow 0}{\sim} \frac{16}{d + 4\bar{Q}} u^{(d+4\bar{Q})/16}, \quad a(u) \underset{\tau \rightarrow 0}{\sim} u^{d/4+2+\bar{Q}/2} \tag{4.18}$$

and then

$$a(\tau) \underset{\tau \rightarrow 0}{\sim} \tau^{4+8(4-\bar{Q})/(d+4\bar{Q})}. \tag{4.19}$$

Since we have a power-law expansion at both limits, the Hubble parameter goes as $H(\tau) \propto 1/\tau$ and more precisely is asymptotically given by:

$$H(\tau) \underset{\tau \rightarrow \infty}{\sim} \frac{2}{\tau} \quad H(\tau) \underset{\tau \rightarrow 0}{\sim} \frac{2}{\tau} \left(1 + \frac{16 + d}{d + 4\bar{Q}}\right). \tag{4.20}$$

Note that this behaviour precisely measures the effect of a finite value for \bar{Q} and in fact for $\bar{Q} \rightarrow \infty$ we recover again $H(\tau) \sim 2/\tau$. To summarize, we get again a polynomially expanding universe (a so-called big-bang solution).

Damped oscillation regime. The small- \bar{Q} regime is more difficult to be studied analytically. Nevertheless we can make some qualitative comments on the overall behaviour of the solution.

¹¹The hypergeometric function ${}_2F_1$ is defined as follows:

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{4.14}$$

where $(a)_k$ is the Pochhammer symbol

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}. \tag{4.15}$$

First of all we can redefine the variables so to recast eqs. (3.22) in the form

$$\sigma(t) = Ae^{-\bar{Q}t} \cos(\omega t) \quad (4.21a)$$

$$\Phi(t) = \Phi_0 + \frac{Ad}{2} e^{-\bar{Q}t} \cos(\omega t) - \bar{Q}t \quad (4.21b)$$

where A is a constant and $\omega^2 = |\bar{Q}^2 - 4R/d|$. The warp factor and the Hubble parameter then read

$$a(\tau) = e^{2\sigma(t) - \Phi(t)/2} \Big|_{t=t(\tau)} = \exp \left[Ae^{-\bar{Q}t} \cos(\omega t) \left(2 - \frac{d}{4} \right) - \frac{\Phi_0}{2} + \frac{\bar{Q}t}{2} \right] \Big|_{t=t(\tau)}, \quad (4.22)$$

$$H(\tau) = e^{\Phi(t)/4} \left(2\dot{\sigma}(t) - \frac{\dot{\Phi}(t)}{2} \right) \Big|_{t=t(\tau)} = \exp \left[\frac{\Phi_0}{4} + \frac{Ad}{8} e^{-\bar{Q}t} \cos(\omega t) - \frac{\bar{Q}t}{4} \right] \times \\ \times \left(-Ae^{-\bar{Q}t} (\bar{Q} \cos(\omega t) + \omega \sin(\omega t)) \left(2 - \frac{d}{4} \right) + \frac{\bar{Q}}{2} \right) \Big|_{t=t(\tau)}, \quad (4.23)$$

where $\tau(t)$ is defined by

$$\tau(t) = \int_0^t e^{-\Phi(t')/4} dt' = \int_0^t \exp \left[-\frac{\Phi_0}{4} - \frac{Ad}{8} e^{-\bar{Q}t'} \cos(\omega t') + \frac{\bar{Q}t'}{4} \right] dt'. \quad (4.24)$$

The Einstein frame time variable $\tau(t)$ is given by the integral of a positive function of t . This means that it is monotonically increasing and the relation $\tau = \tau(t)$ is invertible. In turn the qualitative behaviour of the functions $a = a(\tau)$ and $H = H(\tau)$ can be understood by studying their behaviours as functions of the string frame time t . In particular $H(\tau)$ is an oscillating function and has a zero for each zero of the function $(2\dot{\sigma}(t) - \dot{\Phi}(t)/2)$. The maximum number \bar{N} of such zeros can be estimated in the following way. If we write explicitly the equation $2\dot{\sigma} - \dot{\Phi}/2 = 0$ we obtain

$$\bar{Q} = 2Ae^{-\bar{Q}t} (\bar{Q} \cos(\omega t) + \omega \sin(\omega t)) \left(2 - \frac{d}{4} \right), \quad (4.25)$$

so, we are looking for the intersections of the line $\varpi(t) = \bar{Q}$ with the function $\varpi(t) = 2Ae^{-\bar{Q}t} (\bar{Q} \cos(\omega t) + \omega \sin(\omega t)) \left(2 - \frac{d}{4} \right)$. The latter is bounded by

$$\varpi(t) = 2Ae^{-\bar{Q}t} \sqrt{\bar{Q}^2 + \omega^2} \left| 2 - \frac{d}{4} \right| \quad (4.26)$$

and then intersections are only possible for $t < \bar{t}$ where \bar{t} is given by

$$\bar{t} = \frac{1}{\bar{Q}} \log \frac{A\sqrt{\bar{Q}^2 + \omega^2} |8 - d|}{2\bar{Q}}. \quad (4.27)$$

In the $(0, \bar{t})$ interval, the $(\bar{Q} \cos(\omega t) + \omega \sin(\omega t))$ term can oscillate $\omega\bar{t}/(2\pi)$ times and then the number of solutions for eq. (4.25) is at most

$$n < \bar{N} = \left\lceil \frac{\omega\bar{t}}{\pi} \right\rceil = \left\lceil \frac{\omega}{\bar{Q}\pi} \log \frac{A\sqrt{\bar{Q}^2 + \omega^2} |8 - d|}{2\bar{Q}} \right\rceil. \quad (4.28)$$

The overall behaviour for the warp factor $a(\tau)$ is of that of a power-law expansion superimposed with $n < \bar{N}$ oscillations. The Hubble parameter after n oscillations goes to zero for large τ as $H(\tau) \underset{\tau \rightarrow \infty}{\sim} 2/\tau$. Typical behaviours for $a(\tau)$ and $H(\tau)$ are shown in figure 3.

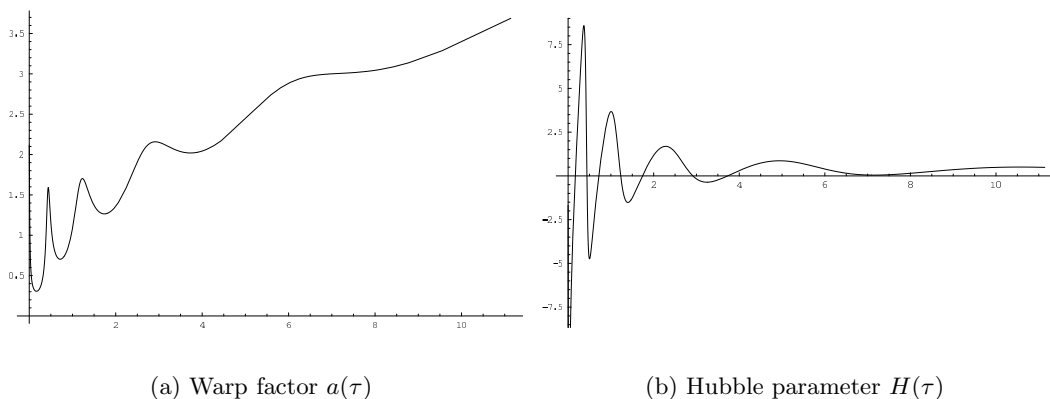


Figure 3: Typical behaviour for the warping factor and Hubble parameter in the small- \bar{Q} regime. Numerical solution with parameters $A = 1, \bar{Q} = 1.2, \omega = 10, d = 3$.

5. Summary

Finding viable time-dependent solutions with cosmological interest is one of the open problems in string theory. In this note we have presented family of such solutions, derived as deformations of a conformal model, leading to a generalization of FRW universe undergoing a big bang-like expansion, superposed with damped oscillations. In these backgrounds the time dependence comes from a Liouville-type mechanism in which flowing from the conformal point is seen as the effect of a time-dependent field. The RG-flow equations on constant-time slices then become an approximation for the Weyl-invariance equations and the off-shell dynamics is interpreted as the on-shell dynamics with one extra dimension. As already stressed, this behaviour is not generic and the identification of the RG-flow with time evolution is not valid in any regime.

The constraints imposed by the rational nature of the underlying compact model essentially fix the type of deformations one can consider and naturally lead to an isotropic FRW background whose behaviour is due to the presence of an axion (dual to the Kalb-Ramond field) and an oscillating dilaton. Anisotropic extensions of FRW solution are also considered but time evolution smoothes the initial anisotropy and the system relaxes to a symmetric phase.

A common feature of all situations considered here is that the RG equations at lowest order in α' always describe a Ricci flow with torsion. In the case of the three-sphere with anisotropic perturbation, the Ricci-flow equations are identical to the Halphen system, and are thereby integrable by using modular forms. More general perturbations exist, which are compatible with the mini-superspace approximation (see [20]) but a systematic analysis of all those is beyond the scope of this work.

It stems from the overall analysis performed in the present framework that in a quite universal manner, sooner or later, time evolution is identified with an RG flow, which is a Ricci flow. Time becomes thus a two-dimensional scale, the dilaton being the interplay

between target space and worldsheet. Thurston’s geometrization conjecture is then at work. It tells us that the string target space will universally converge towards a collection of Einstein spaces, whenever these are available, enhancing therefore the symmetry. This behaviour is independent of the initial conditions.

Clearly, this reasoning is valid under the assumption that the initial space is locally homogeneous. Exact time-dependent string backgrounds with non-homogeneous spaces exist [40, 41], and their time evolution does not lead to Einstein spaces. Consistently with the above picture, in these cases, time evolution never identifies with an RG flow because it is rather related to a marginal deformation.

The latter observations are puzzling additions to the already mysterious origin of time in string theory, along the lines of thought of A. Polyakov.

Acknowledgments

The authors thank C. Bachas, T. Damour, G. Gibbons, P. Grange, C. Kounnas, J. Rizos and K. Sfetsos for stimulating scientific discussions. This work was supported in part by the EU under the contracts MEXT-CT-2003-509661, MRTN-CT-2004-005104, MRTN-CT-2004-503369. Domenico Orlando is supported by the European Commission FP6 RTN programme MRTN-CT-2004-005104 and in part by the Belgian Federal Science Policy Office through the Interuniversity Attraction Pole P5/27 and in part by the “FWO-Vlaanderen” through project G.0428.06 and acknowledges financial support by the Agence Nationale pour la Recherche, France. Marios Petropoulos acknowledges partial financial support by the Swiss National Science Foundation.

A. The Ricci tensor for the deformed group manifold

In order to calculate $R_{\alpha\beta}$, we need to use some notions in group manifold geometry. Let $\{\hat{\theta}^\alpha\}$ be a set of one-forms on a manifold \mathcal{M} satisfying the commutation relations

$$[\hat{\theta}^\beta, \hat{\theta}^\gamma] = f^\alpha_{\beta\gamma} \hat{\theta}^\alpha \tag{A.1}$$

as it is the case when $\hat{\theta}^\alpha$ are the Maurer-Cartan one-forms for a Lie algebra and $f^\alpha_{\beta\gamma}$ the corresponding structure constants. We wish to study the geometry of the Riemann manifold \mathcal{M} endowed with the metric

$$g = g_{\alpha\beta} \hat{\theta}^\alpha \otimes \hat{\theta}^\beta. \tag{A.2}$$

In general such a metric has a symmetry $G \times G'$ where G is the group corresponding to the structure constants $f^\alpha_{\beta\gamma}$ and $G' \subset G$. The maximally symmetric case, in which $G' = G$ is obtained when g is G -invariant, i.e. when it satisfies

$$f^\alpha_{\beta\gamma} g_{\alpha\delta} + f^\alpha_{\delta\gamma} g_{\alpha\beta} = 0. \tag{A.3}$$

For compact groups this condition is fulfilled by the Killing metric.

The connection one-forms ω^α_β are uniquely determined by the compatibility condition and the vanishing of the torsion. Respectively:

$$dg_{\alpha\beta} - \omega^\gamma_\alpha g_{\gamma\beta} - \omega^\gamma_\beta g_{\gamma\alpha} = 0, \tag{A.4}$$

$$d\hat{\theta}^\alpha + \omega^\alpha_\beta \wedge \hat{\theta}^\beta = T^\alpha = 0. \tag{A.5}$$

As it is shown in [42], if $g_{\alpha\beta}$ is constant, the solution to the system can be put in the form

$$\omega^\alpha_\beta = -D^\alpha_{\beta\gamma} \hat{\theta}^\gamma \tag{A.6}$$

where $D^\alpha_{\beta\gamma} = 1/2 f^\alpha_{\beta\gamma} - K^\alpha_{\beta\gamma}$ and $K^\alpha_{\beta\gamma}$ is a tensor (symmetric in the lower indices) given by:

$$K^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\kappa} f^\delta_{\kappa\beta} g_{\gamma\delta} + \frac{1}{2} g^{\alpha\kappa} f^\delta_{\kappa\gamma} g_{\beta\delta}. \tag{A.7}$$

The Riemann tensor and the corresponding Ricci tensor now read:

$$R^\alpha_{\beta\gamma\delta} = D^\alpha_{\beta\kappa} f^\kappa_{\gamma\delta} + D^\alpha_{\kappa\gamma} D^\kappa_{\beta\delta} - D^\alpha_{\kappa\delta} D^\kappa_{\beta\gamma} \tag{A.8}$$

$$R_{\beta\delta} = D^\alpha_{\beta\kappa} f^\kappa_{\alpha\delta} - D^\alpha_{\kappa\delta} D^\kappa_{\beta\alpha}. \tag{A.9}$$

B. The Jacobi functions

We take the standard conventions for the Jacobi functions:

$$\vartheta_2(z) = \sum_{p \in \mathbb{Z}} q^{\frac{1}{2}(p+\frac{1}{2})^2}, \tag{B.1}$$

$$\vartheta_3(z) = \sum_{p \in \mathbb{Z}} q^{\frac{p^2}{2}}, \tag{B.2}$$

$$\vartheta_4(z) = \sum_{p \in \mathbb{Z}} (-1)^p q^{\frac{p^2}{2}}, \tag{B.3}$$

where $q = \exp 2i\pi z$. With these conventions

$$\lambda = \frac{\vartheta_2^4}{\vartheta_3^4}, \quad 1 - \lambda = \frac{\vartheta_4^4}{\vartheta_3^4} \tag{B.4}$$

and

$$2\eta^3 = \vartheta_2 \vartheta_3 \vartheta_4 \tag{B.5}$$

with

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \tag{B.6}$$

the Dedekind function. The non-holomorphic function

$$\hat{E}(z) = \frac{12}{i\pi} \frac{d \log \eta}{dz} - \frac{3}{\pi \text{Im } z} \tag{B.7}$$

is modular covariant of degree 2. The holomorphic part of the latter, which is not modular-covariant, is the function y (divided by $i\pi$) in (2.31).

References

- [1] F. David, *Conformal field theories coupled to 2-d gravity in the conformal gauge*, *Mod. Phys. Lett. A* **3** (1988) 1651.
- [2] J. Distler and H. Kawai, *Conformal field theory and 2-d quantum gravity or who's afraid of Joseph Liouville?*, *Nucl. Phys. B* **321** (1989) 509.
- [3] S.R. Das, A. Dhar and S.R. Wadia, *Critical behavior in two-dimensional quantum gravity and equations of motion of the string*, *Mod. Phys. Lett. A* **5** (1990) 799.
- [4] A. Sen, *Nontrivial renormalization group fixed points and solutions of string field theory equations of motion*, *Phys. Lett. B* **252** (1990) 566.
- [5] A.R. Cooper, L. Susskind and L. Thorlacius, *Two-dimensional quantum cosmology*, *Nucl. Phys. B* **363** (1991) 132.
- [6] M. Carfora and K. Piotrkowska, *A renormalization group approach to relativistic cosmology*, *Phys. Rev. D* **52** (1995) 4393 [[gr-qc/9502021](#)].
- [7] B. Chow and D. Knopf, *The Ricci flow: an introduction*, *Mathematical Surveys and Monographs* **110** (2004) 630.
- [8] T. Suyama, *Closed string tachyon condensation in supercritical strings and rg flows*, *JHEP* **03** (2006) 095 [[hep-th/0510174](#)].
- [9] M. Headrick and T. Wiseman, *Ricci flow and black holes*, *Class. and Quant. Grav.* **23** (2006) 6683 [[hep-th/0606086](#)].
- [10] K. Graham, A. Konechny and J. Teschner, *On the time-dependent description for the decay of unstable D-branes*, [hep-th/0608003](#).
- [11] T. Damour and A.M. Polyakov, *The string dilaton and a least coupling principle*, *Nucl. Phys. B* **423** (1994) 532 [[hep-th/9401069](#)].
- [12] M. Gasperini, F. Piazza and G. Veneziano, *Quintessence as a run-away dilaton*, *Phys. Rev. D* **65** (2002) 023508 [[gr-qc/0108016](#)].
- [13] I. Antoniadis, C. Bachas, J.R. Ellis and D.V. Nanopoulos, *Cosmological string theories and discrete inflation*, *Phys. Lett. B* **211** (1988) 393.
- [14] I. Antoniadis, C. Bachas, J.R. Ellis and D.V. Nanopoulos, *An expanding universe in string theory*, *Nucl. Phys. B* **328** (1989) 117.
- [15] L. Alvarez-Gaumé, D.Z. Freedman and S. Mukhi, *The background field method and the ultraviolet structure of the supersymmetric nonlinear sigma model*, *Ann. Phys. (NY)* **134** (1981) 85.
- [16] E. Braaten, T.L. Curtright and C.K. Zachos, *Torsion and geometrostatics in nonlinear sigma models*, *Nucl. Phys. B* **260** (1985) 630.
- [17] D.H. Friedan, *Nonlinear models in two + epsilon dimensions*, *Ann. Phys. (NY)* **163** (1985) 318.
- [18] H. Osborn, *General bosonic sigma models and string effective actions*, *Ann. Phys. (NY)* **200** (1990) 1.
- [19] V.A. Fateev, E. Onofri and A.B. Zamolodchikov, *The sausage model (integrable deformations of $O(3)$ sigma model)*, *Nucl. Phys. B* **406** (1993) 521.

- [20] V.A. Fateev, *The sigma model (dual) representation for a two-parameter family of integrable quantum field theories*, *Nucl. Phys. B* **473** (1996) 509.
- [21] J. Isenberg and M. Jackson, *Ricci flow of locally homogeneous geometies on closed manifolds*, *J. Differential Geom.* **35** (1992) 723.
- [22] G. Darboux, *Mmoire sur la théorie des coordonnées curvilignes et des systmes orthogonaux*, *Ann. Ec. Normale Supr.* **7** (1878) 101.
- [23] G.-H. Halphen, *Sur un système d'équations différentielles*, *C.R. Acad. Sc. Paris* **92** (1881) 1001.
- [24] G.-H. Halphen, *Sur certains systèmes d'équations différentielles*, *C.R. Acad. Sc. Paris* **92** (1881) 1004.
- [25] G.W. Gibbons and C.N. Pope, *The positive action conjecture and asymptotically euclidean metrics in quantum gravity*, *Commun. Math. Phys.* **66** (1979) 267.
- [26] M.F. Atiyah and N.J. Hitchin, *Low-energy scattering of nonabelian monopoles*, *Phys. Lett. A* **107** (1985) 21.
- [27] J. Chazy, *Sur les équations différentielles dont l'intégrale possde une coupure essentielle mobile*, *C.R. Acad. Sc. Paris* **150** (1910) 456.
- [28] J. Chazy, *Sur les équations différentielles du trisième et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes*, *Acta Math.* **34** (1911) 317.
- [29] L.A. Takhtajan, *A simple example of modular forms as tau functions for integrable equations*, *Theor. Math. Phys.* **93** (1992) 1308.
- [30] L. Ford, *Automorphic functions*, *Chelsea Pub. Co.* (1951).
- [31] J.-P. Serre, *Cours d'arithmétique*, *Presses Universitaires de France, Paris* (1970).
- [32] G.W. Gibbons and N.S. Manton, *Classical and quantum dynamics of BPS monopoles*, *Nucl. Phys. B* **274** (1986) 183.
- [33] A.M. Polyakov, *The wall of the cave*, *Int. J. Mod. Phys. A* **14** (1999) 645 [hep-th/9809057].
- [34] J.R. Ellis, N.E. Mavromatos and D.V. Nanopoulos, *A microscopic Liouville arrow of time*, hep-th/9805120.
- [35] J.R. Ellis, N.E. Mavromatos and D.V. Nanopoulos, *How large are dissipative effects in noncritical Liouville string theory?*, *Phys. Rev. D* **63** (2001) 024024 [gr-qc/0007044].
- [36] A.A. Tseytlin, *Dilaton, winding modes and cosmological solutions*, *Class. and Quant. Grav.* **9** (1992) 979 [hep-th/9112004].
- [37] A.A. Tseytlin, *Cosmological solutions with dilaton and maximally symmetric space in string theory*, *Int. J. Mod. Phys. D* **1** (1992) 223 [hep-th/9203033].
- [38] D.S. Goldwirth and M.J. Perry, *String dominated cosmology*, *Phys. Rev. D* **49** (1994) 5019 [hep-th/9308023].
- [39] E.J. Copeland, A. Lahiri and D. Wands, *Low-energy effective string cosmology*, *Phys. Rev. D* **50** (1994) 4868 [hep-th/9406216].
- [40] P.M. Petropoulos and K. Sfetsos, *Non-Abelian coset string backgrounds from asymptotic and initial data*, hep-th/0610055.

- [41] P.M. Petropoulos, *Parafermions, brane distributions and FRW hierarchies*, hep-th/0612243.
- [42] F. Mueller-Hoissen and R. Stuckl, *Coset spaces and ten-dimensional unified theories*, *Class. and Quant. Grav.* **5** (1988) 27.